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Hartley Proper Efficiency In Multiobjective Optimization Problems With Locally Lipschitz Set-Valued Objectives and Constraints

P. H. SACH
Hanoi Institute of Mathematics, 18 Hoang Quoc Viet Road, 10307 Hanoi Vietnam (e-mail: phsach@math.ac.vn)

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Abstract. In this paper we give necessary conditions for Hartley proper efficiency in a vector optimization problem whose objectives and constraints are described by nonconvex locally Lipchitz set-valued maps. The obtained necessary conditions are written in terms of a Lagrange multiplier rule. Our approach is based on a reduction theorem which leads the problem of studying proper efficiency to a scalar optimization problem whose objective is given by a function of max-type. Sufficient conditions for Hartley proper efficiency are also considered.

Key words: Clarke subdifferential, locally Lipschitz set-valued map, proper efficiency, vector optimization problem.

1. Introduction

Proper efficiencies are notions used in vector optimization to exclude anomalous efficient solutions. Such notions are introduced and studied by Kuhn-Tucker [17], Geoffrion [10], Borwein [3,4], Benson [2], Hartley [13], Henig [14], Borwein-Zhuang [5,6] ... The reader is referred to [12] for a comprehensive survey of proper efficiencies and relationships between them. Characterizations of proper efficiencies are obtained in [2–5,10,24] for convex vector optimization problems and in [15,18] for the nonconvex case. Recently, some of them are extended [16,19–22,26,27] to problems with set-valued objectives and constraints being convex in a generalized sense: Benson proper efficiency is characterized [19,22,26,27] in terms of minimizers of a suitable scalar optimization problem, Lagrange multipliers and saddle points under a subconvexlikeness assumption [19] or near subconvexlineness assumption [22,27] (see also [26]). Super efficiency is expressed similarly in [21] under a convexlikeness assumption and in [20] under a near convexlikeness assumption. Hartley proper efficiency is characterized [16] by a scalarization theorem, provided that set-valued maps involved in objectives and constraints of vector optimization problems are nearly subconvexlike, or their support functions are quasiconvex. Henig proper efficiency in set-valued optimization is studied in [11] by means of epiderivatives [11] of set-valued maps.

In this paper we are interested in Hartley proper efficiency for problems which may not be convex but can be described by locally Lipschitz set-valued maps. Our main idea, based on Reduction Theorem 2.1 of Section 2, is to prove that the problem of finding properly efficient points is equivalent to that of a scalar optimization problem whose objective function is a function of max-type. (Such an idea can be found e.g., in [15,18] for the single-valued case.) Since this equivalence is established without any requirement imposed on the data of vector optimization problems, our scalarizing approach can be applied to various classes of problems. The class considered in this paper consists of problems which are given by locally Lipschitz set-valued maps. Under suitable assumptions we express necessary conditions for Hartley proper efficiency in terms of the Clarke subdifferentials of some real-valued functions which are constructed from the data of problems under consideration. These results are given in Section 4 and are obtained by combining the above scalarizing approach with necessary conditions for optimality of a scalar optimization problem considered in Section 3. We also point out that necessary conditions formulated in Section 4 become sufficient conditions for Hartley proper efficiency if some generalized convexity assumptions are satisfied. It is worth noticing that, although we restrict ourselves to the proper efficiency of Hartley, our results are valid for all the proper efficiencies of Benson [2], Borwein [4, Def. 2(b)], Henig [14, Def. 2.1] and, in particular, for a recent notion of proper efficiency of Borwein-Zhuang [6]. This is because the cone D which is used to define proper efficiency in this paper is assumed to be closed convex and pointed; and hence all the just mentioned notions of proper efficiency are equivalent (see [12]). Thus, a comparison of the results of the present paper dealing with Hartley proper efficiency and the results of [11] dealing with globally proper efficiency of Henig [14] is justified since these notions of proper efficiency are equivalent. Let us give such a brief comparison. We begin by a remark that necessary conditions for globally proper efficiency [14] are written in [11] in terms of Y-epiderivatives (see [11, Definition 2.7]) or Clarke tangent epiderivatives (see [11, Definition 2.5]) of arbitrary set-valued maps between normed spaces. Since Y-epiderivatives (resp. Clarke tangent epiderivatives) are defined as set-valued maps whose graphs are cones which contain the Bouligand tangent cones (resp. which coincide with the Clarke tangent cones) of the epigraphs of given set-valued maps, applying the results of [11] requires to find such tangent cones. This seems to be a task not easy even in case when all maps are locally Lipschitz. It is then natural to ask if these epiderivatives can be replaced by other objects in writing necessary conditions for proper efficiency. In this paper we show that this is possible if we restrict ourselves to locally Lipschitz maps between Euclidean spaces and if we use the Clarke subdifferentials of the support functions of these set-valued maps as substitutes for their epiderivatives. This is a difference between the approach of [11] and that of the present paper. Another difference is that the results of [11] are valid only if the cone involved in the constraint of optimization problems (i.e., the cone E in the constraint (3.1) introduced in Section 3) must have nonempty interior while our results are established without this property. Some examples illustrating our main results can be found in Section 5 of this paper.

We conclude our introduction by giving some notions of Nonsmooth Analysis [7] which will be used later.

In this paper all the spaces X, Y and Z are assumed to be finite-dimensional Euclidean spaces. The scalar product of two vectors ξ and x of X can be written as $\xi^t x$ where t denotes the transpose. The unit sphere and the closed unit ball of X are denoted by S_X and B_X, respectively. For the sake of simplicity we will delet the subscript X in the symbols S_X and B_X. To show that Euclidean space X is m-dimensional we write $X = R^m$. The nonnegative orthant of R^m is denoted by R^m_+.

The graph of a set-valued map $F: X \rightrightarrows Y$ is denoted by $\mathrm{gr}\, F := \{(x, y) : y \in F(x)\}$. Set-valued map F is said to be closed if $\mathrm{gr}\, F$ is a closed set. If for each $x \in X$ the set $F(x)$ is closed (resp. convex, compact) then we say that F has closed (resp. convex, compact) values. For each point $\zeta \in Y$ we introduce the extended function $c_F(\zeta, \cdot) : X \to R$ (the real line) defined by

$$c_F(\zeta, x) = \inf\{\zeta^t y : y \in F(x)\}, \quad x \in X,$$

where we set $c_F(\zeta, x) = +\infty$ if $F(x) = \emptyset$ (the empty set). In case when F is locally Lipschitz and has compact values, $c_F(\zeta, \cdot)$ is a locally Lipschitz function, and for each $x_0 \in X$ there exists a neighbourhood of x_0 such that $c_F(\zeta, \cdot)$ admits $\alpha\|\zeta\|$ as its Lipschitz constant at x_0 where α is a Lipschitz constant of F at x_0. Recall that set-valued map F is locally Lipschitz at $x_0 \in X$ with Lipschitz constant $\alpha > 0$ if there exists a neighbourhood U of x_0 such that

$$F(x_1) \subset F(x_2) + \alpha\|x_1 - x_2\| B_Y, \quad \forall x_1, x_2 \in U.$$

We say that F is locally Lipschitz if it is locally Lipschitz at any point $x_0 \in X$.

A locally Lipschitz function $f: X \to R$ is said to be regular in the sense of Clarke [7] at $x_0 \in X$ if the one-sided directional derivative

$$f'(x_0, x) := \lim_{\lambda \downarrow 0} \lambda^{-1}\left[f(x_0 + \lambda x) - f(x_0)\right]$$

exists and equals the Clarke directional derivative

$$f^0(x_0, x) := \limsup_{x' \to x_0, \lambda \downarrow 0} \lambda^{-1}\left[f(x' + \lambda x) - f(x')\right]$$

for each $x \in X$. The (Clarke) subdifferential of locally Lipschitz function f at x_0 is denoted by

$$\partial f(x_0) := \{\xi \in X : f^0(x_0, x) \geqslant \xi^t x \ \ \forall x \in X\}.$$

LEMMA 1.1 *(see [7, Proposition 2.3.3 and 2.3.12]). Let* $f_1, f_2, \ldots, f_m$ *be locally Lipschitz funtions. Let* $f(\cdot) = f_1(\cdot) + f_2(\cdot) + \cdots + f_m(\cdot)$ *and* $\psi(\cdot) = \max\{f_i(\cdot) : i = 1, 2, \ldots, m\}$. *Then*

$$\partial f(x) \subset \partial f_1(x) + \partial f_2(x) + \cdots + \partial f_m(x),$$
$$\partial \psi(x) \subset \mathrm{co}\,\{\partial f_i(x) : i \in I(x)\},$$

where $I(x) = \{i : 1 \leqslant i \leqslant m,\ f_i(x) = \psi(x)\}$ *and* co *denotes the convex hull.*

For a closed set $C \subset X$ we denote by $\rho_C(x)$ the distance from $x \in X$ to C. The set $T_C(x_0) := \{x \in X : \rho_C^0(x_0, x) \leqslant 0\}$ is the Clarke tangent cone of C at $x_0 \in C$. The set $N_C(x_0) = \{\xi \in X : \xi^t x \leqslant 0 \quad \forall x \in T_C(x_0)\}$ is the Clarke normal cone of C at x_0.

For a convex cone $D \subset Y$ let us write

$$D^+ = \{\zeta \in Y : \zeta^t d \geqslant 0 \quad \forall d \in D\},$$
$$D^{+i} = \left\{\zeta \in Y : \zeta^t d > 0 \quad \forall d \in D \setminus \{0\}\right\},$$
$$D_S^+ = D^+ \cap S,\ D_S^{+i} = D^{+i} \cap S,\ D_B^+ = D^+ \cap B,\ D_B^{+i} = D^{+i} \cap B$$

where S and B are the unit sphere and the closed unit ball of Y. We say that D is pointed if $y \in D \cap -D \Rightarrow y = 0$.

2. A Reduction Theorem

Our approach to Hartley proper efficiency for a nonconvex vector set-valued optimization problem consists of three steps:

(i) To prove that the problem of finding a Hartley properly efficient point in problems with set-valued objectives is equivalent to a scalar optimization problem whose single objective is described by a function of max-type.

(ii) To derive necessary optimality conditions for optimizing a real-valued function subject to nonconvex set-valued constraints.

(iii) To combine the results of the first two steps to obtain necessary conditions for Hartley proper efficiency in a vector optimization problem with objectives and constraints being described by set-valued maps.

This section is devoted to the first of the above three steps. Namely, a Reduction Theorem is established to reduce the proper efficiency in vector set-valued optimization to a scalar optimization problem with objective function being a function of max-type. It is worth noticing that the Reduction Theorem 2.1 below is obtained without convexity or continuity of maps involved in the vector optimization problem under consideration.

We first recall some notions of [13]. Let D be a closed convex pointed cone of Y with $D \neq \{0\}$, and $\mathcal{F}$ a subset of Y. A point $y_0 \in \mathcal{F}$ is said to be a D-efficient point of $\mathcal{F}$ if

$$\forall y \in \mathcal{F} : y - y_0 \notin -D \setminus \{0\}. \tag{2.1}$$

A D-efficient point y_0 of $\mathcal{F}$ is said to be a Hartley properly efficient point of $\mathcal{F}$ if there exists a positive number M such that for all $\zeta \in D_S^+$ and $y \in \mathcal{F}$ with $\zeta^t(y - y_0) < 0$ there exists $\widetilde{\zeta} \in D_S^+$ with $\widetilde{\zeta}^t(y - y_0) > 0$ and

$$\frac{\zeta^t(y_0 - y)}{\widetilde{\zeta}^t(y - y_0)} \leqslant M. \tag{2.2}$$

Observe from [12,13] that Hartley proper efficiency and Geoffrion proper efficiency coincide if D is the nonnegative orthant of Y.

Now let $F: X \rightrightarrows Y$ be a set-valued map, Q a nonempty subset of X and D a closed convex pointed cone of Y with $D \neq \{0\}$. Consider the following vector optimization Problem (P):

$$\begin{aligned} &\text{minimize} \quad F(x) \\ &\text{subject to} \quad x \in Q. \end{aligned}$$

Let us fix points $x_0 \in Q$ and $y_0 \in F(x_0)$. We say that (x_0, y_0) is a D-efficient point of (P) if y_0 is a D-efficient point of $\mathcal{F} := F(Q)$. A D-efficient point (x_0, y_0) of (P) is said to be a Hartley properly efficient point of (P) if y_0 is a Hartley properly efficient point of $\mathcal{F} := F(Q)$.

A point (x_0, y_0) is said to be a Benson properly efficient point of (P) (see [2]) if

$$-D \cap \operatorname{cl}\operatorname{cone}\left[F(Q) - y_0 + D\right] = \{0\} \tag{2.3}$$

where $\operatorname{cone} A := \{\lambda a : \lambda \geqslant 0, a \in A\}$ and $\operatorname{cl} A$ denotes the closure of A.

In case F being single-valued, instead of saying that $(x_0, F(x_0))$ is a properly efficient point of (P) (in the sense of Hartley or Benson) we will say that x_0 is a properly efficient point of (P).

The following result is taken from [12] and is needed for later use.

LEMMA 2.1. *Let D be a closed convex pointed cone of Y. Then a point (x_0, y_0) is a Hartley properly efficient point of (P) if and only if it is a Benson properly efficient point of (P).*

Observe that under our assumptions of D the set D^{+i} is nonempty [25]. For $y_0 \in F(x_0)$, $\hat{\zeta} \in D^{+i}$ and a positive number M let us introduce the following function

$$f(x) = \inf_{y \in F(x)} \left[\hat{\zeta}^t (y - y_0) + M \max_{\zeta \in D_B^+} \zeta^t (y - y_0)\right], \quad x \in Q, \tag{2.4}$$

where we set $f(x) = +\infty$ if $F(x) = \emptyset$.

Making use of a minimax theorem we can rewrite the function f as

$$f(x) = \max_{\zeta \in D_B^+} \left[c_F(\hat{\zeta} + M\zeta, x) - (\hat{\zeta} + M\zeta)^t y_0\right] \tag{2.5}$$

if F has convex values.

We now give a set-valued version of Theorem 3.1 of [18].

Reduction Theorem 2.1. A point (x_0, y_0) is a Hartley properly efficient point of (P) if and only if there exist $\hat{\zeta} \in D^{+i}$ and a positive number M such that

$$\min_{x \in Q} f(x) = f(x_0) = 0, \tag{2.6}$$

where $f(x)$ is defined by (2.4).

Proof. The idea of the proof is similar to that of Theorem 3.1 of [18]. (Namely, the argument which is used in the proof of [18, Theorem 3.1] for each point $x \in Q$ is now replaced by a similar argument applied to each point y from the image $F(Q)$ of Q.) So, only the sketch of this proof is given here.

(i) *Necessity.* Observe [25] that D^{+i} is nonempty. Let us take an arbitrary point $\hat{\zeta} \in D^{+i}$ and let us set $\hat{\zeta}_1 = \|\hat{\zeta}\|^{-1}\hat{\zeta}$. Then $\hat{\zeta}_1 \in D_S^{+i} \subset D_S^+$. Let M be the positive number appearing in the definition of Hartley proper efficiency of (x_0, y_0). We will prove the validity of (2.6) where $f(\cdot)$ is defined by (2.4) with $M' := \|\hat{\zeta}\| M$ in place of M. We claim that for each $y \in F(Q)$ there exists a point $\tilde{\zeta} \in D_S^+$ such that

$$\hat{\zeta}^t (y - y_0) + M' \tilde{\zeta}^t (y - y_0) \geqslant 0. \tag{2.7}$$

Indeed, if $\hat{\zeta}^t (y - y_0) \geqslant 0$ then $\tilde{\zeta} = \hat{\zeta}_1$ is the desired point. If $\hat{\zeta}^t (y - y_0) < 0$, i.e., $\hat{\zeta}_1^t (y - y_0) < 0$ then by the proper efficiency of (x_0, y_0) there exists

$\tilde{\zeta} \in D_S^+$ satisfying (2.2) with $\hat{\zeta}_1$ instead of ζ. From this we derive (2.7), as required. Clearly, (2.7) implies that

$$\hat{\zeta}^t(y - y_0) + M' \max_{\zeta \in D_B^+} \zeta^t(y - y_0) \geqslant 0.$$

Since this is true for each $y \in F(Q)$ we obtain $f(x) \geqslant 0$ for all $x \in Q$. In particular, for $x = x_0 \in Q$ we have $f(x_0) \geqslant 0$. On the other hand, $f(x_0) \leqslant 0$ since $y_0 \in F(x_0)$. Therefore, $f(x_0) = 0$. Condition (2.6) is thus established.

(ii) *Sufficiency.* By Lemma 2.1 it suffices to show that (x_0, y_0) is a Benson properly efficient point. We delete the detailed proof of this fact, noting that it can be established by modifying the corresponding argument in the proof of Theorem 3.1 of [18]. □

Before giving a corollary of Theorem 2.1 which extends Theorem 3.1 of [15] let us set

$$v(y) = \max(y^1, y^2, \ldots, y^m, 0)$$

where $y \in Y = R^m$ and $y^i, i = 1, 2, \ldots, m$, are the components of y. We also need the following extended function $\overline{f}: X \to R$ defined by

$$\overline{f}(x) = \inf_{y \in F(x)} [\hat{\zeta}^t(y - y_0) + Mv(y - y_0)], \quad x \in X, \tag{2.8}$$

where we set $\overline{f}(x) = +\infty$ if $F(x) = \emptyset$. Obviously, $\overline{f}$ depends on $\hat{\zeta}$, M and y_0.

COROLLARY 2.1. *Let $D = R_+^m \subset Y = R^m$ and let $\hat{\zeta} \in \text{int } R_+^m$. Then (x_0, y_0) is a Hartley properly efficient point (or, equivalently, a Geoffrion properly efficient point) of (P) if and only if there exists a positive number M such that*

$$\min_{x \in Q} \overline{f}(x) = \overline{f}(x_0) = 0, \tag{2.9}$$

where $\overline{f}$ is defined by (2.8).

Proof. (i) *Necessity.* We have seen in the proof of Theorem 2.1 that for each $y \in F(Q)$ there exists $\tilde{\zeta} \in D_S^+$ satisfying (2.7). Let $\tilde{\zeta}^i$ (resp. y^i; y_0^i), $i = 1, 2, \ldots, m$, be the components of $\tilde{\zeta}$ (resp. y; y_0). Since $\tilde{\zeta}^i \geqslant 0$ and $\|\tilde{\zeta}\| = 1 \geqslant \tilde{\zeta}^i$ for each $i = 1, 2, \ldots, m$, we get

$$\begin{aligned}\tilde{\zeta}^t(y - y_0) &= \sum_{i=1}^m \tilde{\zeta}^i (y^i - y_0^i) \\ &\leqslant \left(\sum_{i=1}^m \tilde{\zeta}^i\right) \max\{y^1 - y_0^1, y^2 - y_0^2, \ldots, y^m - y_0^m\} \\ &\leqslant mv(y - y_0).\end{aligned} \tag{2.10}$$

Combining (2.7) and (2.10) yields

$$\hat{\xi}^t(y-y_0)+mM'v(y-y_0)\geqslant 0.$$

Since this is true for arbitrarily chosen $y\in F(Q)$ we obtain

$$\inf_{x\in Q}\overline{f}(x)\geqslant 0=\overline{f}(x_0)$$

where $\overline{f}(x)$ is defined by (2.8) with mM' in place of M. The necessity part is thus established.

(ii) *Sufficiency.* Since $D=R^m_+$ it is clear that the set D^+_B contains the origin of R^m and all the vectors $e_i\in R^m, i=1,2,\ldots,m$, where e_i is the ith unit vector of R^m, i.e., e_i is the vector of R^m whose ith component equals 1 and other components equal 0. Therefore, $f(x)\geqslant\overline{f}(x)$ for all $x\in Q$. From this and from (2.9) it follows that $f(x)\geqslant 0$ for all $x\in Q$. We have seen in the proof of Theorem 2.1 that this result implies (2.6). To complete our proof it remains to apply Theorem 2.1. □

REMARK 2.1. *Observe that Corollary 2.1 is exactly Theorem 3.1 of [15] if F is single-valued.*

COROLLARY 2.2. *Let $x_0\in Q$ and $y_0\in F(x_0)$. If there exists $\hat{\xi}\in D^{+i}$ such that for all $x\in Q$*

$$c_F(\hat{\xi},x)\geqslant\hat{\xi}^t y_0 \tag{2.11}$$

then (x_0,y_0) is a Hartley properly efficient point of (P).

Proof. We derive from (2.11) that $\hat{\xi}^t(y-y_0)\geqslant 0, \forall y\in F(Q)$. Therefore,

$$\max_{\xi\in D^+_B}\xi^t(y-y_0)\geqslant\|\hat{\xi}\|^{-1}\hat{\xi}^t(y-y_0)\geqslant 0, \forall y\in F(Q).$$

From this we obtain (2.6) where f is defined by (2.4) with $M=1$. To complete our proof it remains to apply Theorem 2.1. □

COROLLARY 2.3. *Let $x_0\in Q$ and $\hat{\xi}\in D^{+i}$ be such that $c_F(\hat{\xi},\cdot)$ attains its minimum on Q at x_0. Then for each $y_0\in F(x_0)$ such that $c_F(\hat{\xi},x_0)=\hat{\xi}^t y_0$ the point (x_0,y_0) is a Hartley properly efficient point of (P).*

REMARK 2.2. *Corollary 2.3 generalizes Theorem 6.2 of Hartley [13]. Corollary 2.2 was proven by Geoffrion in Theorem 1 of [10] for the case when F is single-valued and D is the nonpositive orthant.*

COROLLARY 2.4. *Let F be single-valued. Then x_0 is a Hartley properly efficient point of (P) if and only if there exist $\hat{\zeta} \in D^{+i}$ and $M > 0$ such that x_0 is a minimizer of function*

$$f(x) := \max_{\zeta \in D_B^+} (\hat{\zeta} + M\zeta)^t (F(x) - F(x_0))$$

on the set Q.

3. A Scalar Optimization Problem

This section is devoted to the second of the three steps mentioned in the beginning of the previous section.

In this paper all Clarke subdifferentials are with respect to x. If ω is a parameter and $g(w, \cdot) : X \to R$ is a locally Lipschitz function then we use the symbol $\partial g(w, x)$ to denote the Clarke subdifferential of $g(w, \cdot)$ at x. In other words, $\partial g(w, x) = \partial g(w, \cdot)(x)$. We will need an estimation of the Clarke subdifferential of a functions of max-type.

LEMMA 3.1. *[8,9] Let Ω be a compact topological space and $g : \Omega \times X \to R$ be a function such that*

(i) For each $x \in X g(\cdot, x)$ is upper semicontinuous and for each $\omega \in \Omega g(\omega, \cdot)$ is locally equi-Lipschitz in the sense that for each point $x_0 \in X$ there exist $\alpha > 0$ and $\beta > 0$ such that

$$\|x_i - x_0\| < \alpha, i = 1, 2, \omega \in \Omega \Rightarrow |g(\omega, x_1) - g(\omega, x_2)| \leqslant \beta \|x_1 - x_2\|.$$

(ii) Set-valued map $(\omega, x) \in \Omega \times X \mapsto \partial g(\omega, x) \subset X$ is closed.
Then $\psi(x) := \max_{\omega \in \Omega} g(\omega, x)$ is a locally Lipschitz function and

$$\partial \psi(x) \subset \text{co} \bigcup_{\omega \in W(x)} \partial g(\omega, x)$$

where

$$W(x) = \{\omega \in \Omega : \psi(x) = g(\omega, x)\}.$$

From now on we assume that

$$Q = \{x \in C : G(x) \cap -E \neq \emptyset\} \tag{3.1}$$

where $C \subset X$ is a closed set, $G : X \rightrightarrows Z$ is a locally Lipschitz set-valued map with nonempty compact convex values, and $E \subset Z$ is a closed convex cone which, unlike [11], is not assumed to have a nonempty interior. Observe that (3.1) becomes equality and inequality constraints if G is single-valued and if E is the Cartesian product of a nonnegative orthant and a trivial cone.

Let $x_0 \in Q$ and

$$\varphi(x) = \max_{\zeta \in D_B^+} (\widehat{\zeta} + \gamma\zeta)^t (h(x) - h(x_0)), \quad x \in Q,$$

where $h : X \to Y$ is a locally Lipschitz (single-valued) map, $\widehat{\zeta}$ is a fixed point of D_B^{+i} and γ is a nonnegative constant.

In this section we will consider the following scalar optimization problem (sP):

$$\begin{array}{ll} \text{minimize} & \varphi(x) \\ \text{subject to} & x \in Q \end{array}$$

where Q is defined by (3.1).

For $x_0 \in C$ and $z_0 \in G(x_0) \cap -E$ let us set

$$E_0^+ = \{\mu \in E^+ : c_G(\mu, x_0) = \mu^t z_0 = 0\}.$$

If G is single-valued then E_0^+ is exactly the set of all $\mu \in E^+$ satisfying the complementarity condition $\mu^t G(x_0) = 0$.

We say that condition (CQ) holds at (x_0, z_0) if

$$\mu \in E_0^+ \setminus \{0\} \Rightarrow 0 \notin \partial c_G(\mu, x_0) + N_C(x_0).$$

It is a simple matter to check that condition (CQ) holds at (x_0, z_0) if and only if

$$\forall \mu \in E_0^+ \setminus \{0\}, \quad \forall \xi \in \partial c_G(\mu, x_0), \ \exists x \in T_C(x_0) : \xi^t x < 0.$$

In practical problems we often deal with Problem (sP) where E is the Cartesian product of closed convex cones E_i of Euclidean spaces Z_i and G is a Cartesian product of locally Lipschitz set-valued maps $G_i : X \rightrightarrows Z_i$ with nonempty compact values $(i = 1, 2)$:

$$\begin{aligned} &E = E_1 \times E_2 \subset Z_1 \times Z_2 = Z, \\ &G(\cdot) = G_1(\cdot) \times G_2(\cdot). \end{aligned}$$

In this case, if $z_0 = (z_{10}, z_{20}) \in Z_1 \times Z_2$ then $z_0 \in G(x_0) \cap -E$ means that $z_{j0} \in G_j(x_0) \cap -E_j \, (j = 1, 2)$. It is clear that $E^+ = E_1^+ \times E_2^+$, $E_0^+ = E_{10}^+ \times E_{20}^+$ where $E_j^+ = (E_j)^+$ and

$$E_{j0}^+ = \{\mu \in E_j^+ : c_{G_j}(\mu, x_0) = \mu^t z_{j0} = 0\} \quad (j = 1, 2)$$

are defined similarly to E^+ and E_0^+. It is easy to verify that in our case condition (CQ) holds at (x_0, z_0) if the following generalized Mangasarian-Fromovitz condition is satisfied:

$$\left.\begin{array}{l} \text{For all } (\mu_1, \mu_2) \in E_{10}^+ \times E_{20}^+, \\ \left[\mu_1 \neq 0 \Rightarrow 0 \notin \partial c_{G_1}(\mu_1, x_0) + N_C(x_0)\right] \\ \text{and} \\ \left[\mu_2 \neq 0 \Rightarrow \forall \xi_j \in \partial c_{G_j}(\mu_j, x_0)(j=1,2), \quad \exists x \in T_C(x_0)\right. \\ \left.\text{such that } \xi_1^t x \leqslant 0 \text{and} \xi_2^t x < 0\right]. \end{array}\right\} \tag{3.2}$$

Consider now a special case of Problem (P) where Q is given by equality and inequality constraints. More precisely, consider the case

$$Q = \{x : G^j(x) = 0 \ \ (j = 1, 2, \ldots, p), G^j(x) \leqslant 0 \ \ (j = p+1, \ldots, p+s)\}$$

where $G^j : X \to R$ are locally Lipschitz functions. This corresponds to the hypothesis that $C = X$, $E_1 = \{0\} \subset R^p$, $E_2 = R_+^s$, $G_1 = (G^1, G^2, \ldots, G^p)$ and $G_2 = (G^{p+1}, G^{p+2}, \ldots, G^{p+s})$. Let $x_0 \in Q$ and let $I(x_0) = \{l : 1 \leqslant l \leqslant s, G^{p+l}(x_0) = 0\}$ be the index set corresponding to the active constraints at x_0. Then (3.2) holds if the following Mangasarian–Fromovitz condition, introduced in [1], is satisfied:

For all $\xi_j \in \partial G^j(x_0)$, $j = 1, 2, \ldots, p$, and

$\overline{\xi}_l \in \partial G^l(x_0)$, $l \in I(x_0)$, vectors

ξ_j, $j = 1, 2, \ldots, p$, are linearly independent,

and there exists $x \in X$such that

$\xi_j^t x = 0$, $j = 1, 2, \ldots, p$, and $\overline{\xi}_l^t x < 0$, $l \in I(x_0)$.

We will assume that the set-valued map

$$(\mu, x) \in E^+ \times X \longmapsto \partial c_G(\mu, x) \subset X \tag{3.3}$$

is closed (i.e., its graph is a closed set). It can be seen that this closedness property is automatically satisfied if G is single-valued. Another example [8] is the case when locally Lipschitz set-valued map G is E-convex in the sense that $G(X) + E$ is a convex set.

THEOREM 3.1. *Let* $x_0 \in C$ *and* $z_0 \in G(x_0) \cap -E$. *Let set-valued map (3.3) be closed. If* x_0 *is a minimizer of Problem (sP) then there exist* $\alpha_0 \in R_+$, $\zeta_0 \in D_B^+$ *and* $\mu_0 \in E_0^+$ *such that* $\alpha_0 + \|\mu_0\| \neq 0$ *and*

$$0 \in \alpha_0 \partial(\widehat{\zeta} + \gamma\zeta_0)^t h(x_0) + \partial c_G(\mu_0, x_0) + N_C(x_0), \tag{3.4}$$

where $\partial \zeta^t h(x_0)$ *denotes the Clarke subdifferential of* $\zeta^t h(\cdot)$ *at* x_0. *In addition,* $\alpha_0 = 1$ *if condition (CQ) holds at* (x_0, z_0).

Proof. Let us fix a point $d_0 \in D \setminus \{0\}$. Then $\widehat{\zeta}^t d_0 > 0$ since $\widehat{\zeta} \in D^{+i}$. Observe that $-d_0 \notin D$ since by assumption $d_0 \in D \setminus \{0\}$ and D is a pointed cone. In other words, $\rho_D(-d_0) > 0$. Observe that

$$\inf_{d \in D} \zeta^t d = \begin{cases} 0 & \text{if } \zeta \in D^+, \\ -\infty & \text{if } \zeta \notin D^+, \end{cases}$$

that is,

$$\inf_{d \in D} \zeta^t (d_0 + d) = \begin{cases} \zeta^t d_0 & \text{if } \zeta \in D^+, \\ -\infty & \text{if } \zeta \notin D^+. \end{cases}$$

This shows that

$$\max_{\zeta \in B} \inf_{d \in D} \zeta^t (d_0 + d) = \max_{\zeta \in D_B^+} \zeta^t d_0.$$

From this it follows that

$$\begin{aligned} \rho_D(-d_0) &= \inf_{d \in D} \|-d_0 - d\| \\ &= \inf_{d \in D} \max_{\zeta \in B} \zeta^t (d_0 + d) \\ &= \max_{\zeta \in B} \inf_{d \in D} \zeta^t (d_0 + d) \\ &= \max_{\zeta \in D_B^+} \zeta^t d_0. \end{aligned} \tag{3.5}$$

Hence, setting $\widehat{d} = \left[(1+\gamma)\rho_D(-d_0)\right]^{-1} d_0$ we have

$$\widehat{\zeta}^t \widehat{d} + \gamma \max_{\zeta \in D_B^+} \zeta^t \widehat{d} \leqslant \max_{\zeta \in D_B^+} \zeta^t \widehat{d} + \gamma \max_{\zeta \in D_B^+} \zeta^t \widehat{d} = 1. \tag{3.6}$$

Now we introduce the following functions:

$$\psi_1(\varepsilon, x) = \max_{\zeta \in D_B^+} (\widehat{\zeta} + \gamma\zeta)^t (h(x) - h(x_0) + \varepsilon \widehat{d}), \tag{3.7}$$

$$\psi_2(\varepsilon, x) = \max_{\mu \in E_B^+} \left[c_G(\mu, x) - \varepsilon \mu^t z_0\right], \tag{3.8}$$

$$p_\varepsilon(x) \quad = \max\left(\psi_1(\varepsilon, x), \psi_2(\varepsilon, x)\right), \;\; x \in X, \tag{3.9}$$

where $\varepsilon \in (0, 1)$ is a parameter.

Observe that for all $\mu \in E^+$

$$c_G(\mu, x_0) \leqslant \mu^t z_0 \leqslant \varepsilon \mu^t z_0$$

since $z_0 \in G(x_0) \cap -E$. This proves that

$$\psi_2(\varepsilon, x_0) \leqslant 0. \tag{3.10}$$

On the other hand, by (3.6)

$$\psi_1(\varepsilon, x_0) = \varepsilon \left(\widehat{\zeta}^t \widehat{d} + \gamma \max_{\zeta \in D_B^+} \zeta^t \widehat{d} \right) \leqslant \varepsilon.$$

This together with (3.9) and (3.10) yields $p_\varepsilon(x_0) \leqslant \varepsilon$.

We now observe that $p_\varepsilon(x) > 0$ for all $x \in C$. Indeed, assume to the contrary that $p_\varepsilon(x) \leqslant 0$ for some $x \in C$. Then, from the very definition of $p_\varepsilon(x)$ we have

$$\psi_2(\varepsilon, x) = \max_{\mu \in E_B^+} \left[c_G(\mu, x) - \varepsilon \mu^t z_0 \right] \leqslant 0 \tag{3.11}$$

and

$$\max_{\zeta \in D_B^+} (\widehat{\zeta} + \gamma \zeta)^t (h(x) - h(x_0)) < \psi_1(\varepsilon, x) \leqslant 0 \tag{3.12}$$

where the first inequality in (3.12) is valid since $(\widehat{\zeta} + \gamma \zeta)^t \widehat{d} > 0$ for all $\zeta \in D_B^+$.

Because of the compactness and convexity of $G(x)$, and the closedness and convexity of E we can derive from (3.11) and a separation argument that $(G(x) - \varepsilon z_0) \cap -E \neq \emptyset$, or, equivalently, $z - \varepsilon z_0 \in -E$ for some $z \in G(x)$. Since $z \in -E + \varepsilon z_0 \subset -E - \varepsilon E \subset -E$ we conclude that $G(x) \cap -E \neq \emptyset$ which together with condition $x \in C$ proves that $x \in Q$. In addition, (3.12) yields $\varphi(x) < \varphi(x_0)$, a contradiction to the optimality of x_0.

From the above discussion we see that

$$p_\varepsilon(x_0) \leqslant \varepsilon + \inf_{x \in C} p_\varepsilon(x).$$

By the Ekeland Variational Principle [7, Theorem 7.5.1] there exists $x_\varepsilon \in C$ such that $\|x_\varepsilon - x_0\| < \sqrt{\varepsilon}$ and the function $p_\varepsilon(\cdot) + \sqrt{\varepsilon} \| \cdot - x_0 \|$ attains its minimum on C at x_0. We will consider this function for all ε so small that it admits some positive number β (independent of ε) as its Lipschitz constant in some neighbourhood of x_0. Then by [7, Proposition 2.4.3] x_ε is also a minimizer of function $p_\varepsilon(\cdot) + \sqrt{\varepsilon} \| \cdot - x_\varepsilon \| + \beta_0 \rho_C(\cdot)$ on C where β_0 is a fixed number greater than β. Making use of [7, Proposition 2.3.2] and Lemma 1.1 we get

$$0 \in \partial p_\varepsilon(x_\varepsilon) + \beta_0 \partial \rho_C(x_\varepsilon) + \sqrt{\varepsilon} B \tag{3.13}$$

(B being the closed unit ball of X). Observe by Lemma 1.1 that

$$\partial p_\varepsilon(x_\varepsilon) \subset \mathrm{co}\left\{\partial \psi_j(\varepsilon, x_\varepsilon) : j \in J(x_\varepsilon)\right\} \tag{3.14}$$

where

$$J(x_\varepsilon) = \left\{j : 1 \leqslant j \leqslant 2,\, \psi_j(\varepsilon, x_\varepsilon) = p_\varepsilon(x_\varepsilon)\right\}.$$

Now we claim that

(i) If $\psi_1(\varepsilon, x_\varepsilon) > 0$ then there exists $\zeta_\varepsilon \in D_B^+$ such that

$$\partial \psi_1(\varepsilon, x_\varepsilon) \subset \partial (\widehat{\zeta} + \gamma \zeta_\varepsilon)^t h(x_\varepsilon). \tag{3.15}$$

(ii) If $\psi_2(\varepsilon, x_\varepsilon) > 0$ then there exists $\mu_\varepsilon \in E_S^+$ such that

$$c_G(\mu_\varepsilon, x_\varepsilon) - \varepsilon \mu_\varepsilon^t z_0 > 0$$

and

$$\partial \psi_2(\varepsilon, x_\varepsilon) \subset \partial\, c_G(\mu_\varepsilon, x_\varepsilon). \tag{3.16}$$

Indeed, if $\psi_2(\varepsilon, x_\varepsilon) > 0$ then there exists the unique element $\mu_\varepsilon \in E^+$ with $\|\mu_\varepsilon\| = 1$ such that

$$0 < \psi_2(\varepsilon, x_\varepsilon) = c_G(\mu_\varepsilon, x_\varepsilon) - \varepsilon \mu_\varepsilon^t z_0,$$

and by Lemma 3.1 condition (3.16) holds. Thus, the claim (ii) is established. To prove (i) observe that

$$\begin{aligned} 0 < \psi_1(\varepsilon, x_\varepsilon) = &\,\widehat{\zeta}^t (h(x_\varepsilon) - h(x_0) + \varepsilon\, \widehat{d}) \\ &+ \gamma \max_{\zeta \in D_B^+} \zeta^t (h(x_\varepsilon) - h(x_0) + \varepsilon \widehat{d}). \end{aligned} \tag{3.17}$$

We infer that

$$\max_{\zeta \in D_B^+} \zeta^t (h(x_\varepsilon) - h(x_0) + \varepsilon \widehat{d}) > 0. \tag{3.18}$$

Indeed, assume to the contrary that

$$0 \geqslant \max_{\zeta \in D_B^+} \zeta^t (h(x_\varepsilon) - h(x_0) + \varepsilon \widehat{d}) \tag{3.19}$$

Then (3.19) yields

$$0 \geqslant \widehat{\zeta}^t (h(x_\varepsilon) - h(x_0) + \varepsilon \widehat{d}) \tag{3.20}$$

since $\widehat{\zeta} \in D_S^{+i} \subset D_B^+$. Multiplying both sides of (3.19) by γ and summing up the obtained inequality and (3.20) we get $0 \geqslant \psi_1(\varepsilon, x_\varepsilon)$, which is impossible.

From (3.18) it follows that there exists the unique element ζ_ε of D_B^+ at which the function $\zeta^t(h(x_\varepsilon) - h(x_0) + \varepsilon \widehat{d})$ of the variable $\zeta \in D_B^+$ attains its maximum. In other words, ζ_ε is the unique element of D_B^+ at which

$$\psi_1(\varepsilon, x_\varepsilon) = (\widehat{\zeta} + \gamma \zeta_\varepsilon)^t (h(x_\varepsilon) - h(x_0) + \varepsilon \widehat{d}).$$

Applying again Lemma 3.1 we get (3.15). Thus, the claim (i) is proven.

Using the claims (i) and (ii) we can show that there exist $\alpha_\varepsilon \in [0, 1], \zeta_\varepsilon \in D_B^+$ and $\mu_\varepsilon \in E_B^+$ such that

$$c_G(\mu_\varepsilon, x_\varepsilon) - \varepsilon \mu_\varepsilon^t z_0 \geqslant 0, \tag{3.21}$$

$$\partial p_\varepsilon(x_\varepsilon) \subset \alpha_\varepsilon \partial (\widehat{\zeta} + \gamma \zeta_\varepsilon)^t h(x_\varepsilon) + (1 - \alpha_\varepsilon) \partial c_G(\mu_\varepsilon, x_\varepsilon), \tag{3.22}$$

$$\alpha_\varepsilon + \|\mu_\varepsilon\| \geqslant 1. \tag{3.23}$$

Indeed, let us consider the following three possible cases:

(a) $J(x_\varepsilon) = \{1\}$, i.e. $\psi_1(\varepsilon, x_\varepsilon) > 0$ and $\psi_2(\varepsilon, x_\varepsilon) = 0$.
(b) $J(x_\varepsilon) = \{2\}$, i.e. $\psi_1(\varepsilon, x_\varepsilon) = 0$ and $\psi_2(\varepsilon, x_\varepsilon) > 0$.
(c) $J(x_\varepsilon) = \{1, 2\}$, i.e. $\psi_1(\varepsilon, x_\varepsilon) > 0$ and $\psi_2(\varepsilon, x_\varepsilon) > 0$.

In case (a) conditions (3.21)–(3.23) hold if we set $\alpha_\varepsilon = 1$ and $\mu_\varepsilon = 0$, and if ζ_ε is the element in claim (i).

In case (b) conditions (3.21)–(3.23) hold if we set $\alpha_\varepsilon = 0$ and $\zeta_\varepsilon = 0$, and if μ_ε is the element in claim (ii).

In case (c) let ζ_ε and μ_ε be the elements in claims (i) and (ii). Then the existence of $\alpha_\varepsilon \in [0, 1]$ such that $\alpha_\varepsilon, \zeta_\varepsilon$ and μ_ε satisfy (3.21)–(3.23) is derived from (3.14).

We have thus proved that (3.21)–(3.23) hold for suitable $\alpha_\varepsilon \in [0, 1], \zeta_\varepsilon \in D_B^+$ and $\mu_\varepsilon \in E_B^+$.

From (3.13) and (3.22) it is clear that

$$0 \in \alpha_\varepsilon \partial (\widehat{\zeta} + \gamma \zeta_\varepsilon) h(x_\varepsilon) + (1 - \alpha_\varepsilon) \partial c_G(\mu_\varepsilon, x_\varepsilon) + \beta_0 \partial \rho_C(x_\varepsilon) + \sqrt{\varepsilon} B. \tag{3.24}$$

Now let us choose a sequence of positive numbers ε tending to zero. Then, by taking subsequences if necessary we may assume that $\alpha_\varepsilon, \zeta_\varepsilon$ and μ_ε are convergent sequences. Denote by α_0, ζ_0 and μ_0' their limits. Then, obviously $\alpha_0 \in [0, 1], \zeta_0 \in D_B^+$ and $\mu_0' \in E_B^+$. In addition, from (3.23)

$\alpha_0+\|\mu_0'\|\geqslant 1$ i.e., $\alpha_0+\|\mu_0'\|\neq 0$. Now letting $\varepsilon\to 0$ in (3.24) and using the closedness of the maps

$$
\begin{aligned}
(\zeta,x)\in D^+\times X &\mapsto \partial\zeta^t h(x),\\
(\mu,x)\in E^+\times X &\mapsto \partial c_G(\mu,x),\\
x\in X &\mapsto \partial\rho_C(x),
\end{aligned}
$$

we obtain (3.4) with $\mu_0=(1-\alpha_0)\mu_0'$. Observe that $\mu_0\in E_0^+$. Indeed, when $\varepsilon\to 0$ (3.21) yields $c_G(\mu_0,x_0)\geqslant 0$ by the upper semicontinuity of $c_G(\cdot,\cdot)$. Combining the just obtained inequality with the following inequalities $0\geqslant \mu_0^t z_0\geqslant c_G(\mu_0,x_0)$ proves that $\mu_0\in E_0^+$, as desired. The first part of Theorem 3.1 is thus established. The second one is obvious. Indeed, by condition (CQ) $\alpha_0\neq 0$ and hence, we may set $\alpha_0=1$. □

COROLLARY 3.1. *Let $x_0\in C$ and $z_0\in G(x_0)\cap -E$. Let set-valued map (3.3) be closed. Let x_0 be a minimizer of the problem of minimizing h subject to $x\in Q$ where $h: X\to R$ is a locally Lipschitz function and Q is defined by (3.1). Then there exist $\alpha_0\in R_+$, $\mu_0\in E_0^+$, not both zero, such that*

$$0\in\alpha_0\partial h(x_0)+\partial c_G(\mu_0,x_0)+N_C(x_0).$$

In addition, $\alpha_0=1$ if condition (CQ) holds at (x_0,z_0).

Proof. This is a special case of Theorem 3.1 where $D=R_+$, $\widehat{\zeta}=1$ and $\gamma=0$.

4. Hartley Proper Efficiency in Vector Set-Valued Optimization Problems

Having a reduction theorem in Section 2 and optimality conditions for a scalar optimization problem in Section 3, we now derive necessary conditions for Hartley proper efficiency in Problem (P) where Q is defined by (3.1). (This is the step (iii) we mentioned in Section 2.) Throughout this section we assume that C is a closed set, D and E are closed convex cones, and $F: X\rightrightarrows Y$ and $G: X\rightrightarrows Z$ are locally Lipschitz set-valued maps with nonempty convex compact values. We additionally assume that $D\neq\{0\}$ and D is pointed. Together with set-valued map (3.3) we also consider the following set-valued map

$$(\zeta,x)\in D^+\times X\longmapsto \partial c_F(\zeta,x)\subset X. \tag{4.1}$$

For $y_0\in Y$ and $z_0\in Z$ let us set

$$D_0^{+i}=\{\zeta\in D^{+i}: c_F(\zeta,x_0)=\zeta^t y_0\}, \tag{4.2}$$
$$E_0^+=\{\mu\in E^+: c_G(\mu,x_0)=\mu^t z_0=0\}. \tag{4.3}$$

Before formulating Theorem 4.1 let us introduce the following invexity notion taken from [24]: locally Lipschitz functions $g_1, g_2, \dots, g_l$ defined on X are called invex on C at $x_0 \in C$ if

$$\forall x \in C, \forall u_i \in \partial g_i(x_0) \ (i=1,2,\dots,l), \quad \exists \eta \in T_C(x_0),$$
$$\forall i = 1,2,\dots,l: g_i(x) - g_i(x_0) \geqslant u_i^t \eta.$$

Characterization of invexity in the just mentioned sense and links with other notions of invexity can be found in [24]. Observe that if C is convex and $g_i (i=1,2,\dots,l)$ are convex then g_i $(i=1,2,\dots,l)$ are invex on C at x_0, with $\eta = x - x_0$.

THEOREM 4.1. *Let $x_0 \in C$, $y_0 \in F(x_0)$ and $z_0 \in G(x_0) \cap -E$. Let set-valued maps (3.3) and (4.1) be closed. If (x_0, y_0) is a Hartley properly efficient point of Problem (P) then there exist a positive integer $k \leqslant 1 + \dim X$, and points $\zeta_j \in D_0^{+i} (j=1,2,\dots,k)$, $\mu \in E_0^+$ and $\alpha_0 \in R_+$ such that $\alpha_0 + \|\mu\| \neq 0$ and*

$$0 \in \alpha_0 \sum_{j=1}^{k} \partial c_F(\zeta_j, x_0) + \partial c_G(\mu, x_0) + N_C(x_0). \tag{4.4}$$

The converse statement is true if $\alpha_0 = 1$ and if functions $c_G(\mu, \cdot)$ and $c_F(\zeta_j, \cdot)$ $(j=1,2,\dots,k)$ are invex on C at x_0.

REMARK 4.1. *If condition (CQ) holds at (x_0, y_0) then the number α_0 appearing in the formulation of necessary conditions for Hartley proper efficiency in Theorem 4.1 must be positive and hence, we can set $\alpha_0 = 1$. The same remark is true for Theorems 4.2 and 4.3 , and all the corollaries of Theorem 4.3.*

Proof of Theorem 4.1. By Theorem 2.1 the function $h(\cdot) = f(\cdot)$ defined by (2.5) must attain its minimum on Q at x_0. By Corollary 3.1 there exist $\mu \in E_0^+$ and $\alpha_0 \in R_+$ such that $\alpha_0 + \|\mu\| \neq 0$ and

$$0 \in \alpha_0 \partial h(x_0) + \partial c_G(\mu, x_0) + N_C(x_0). \tag{4.5}$$

On the other hand, in view of Lemma 3.1

$$\partial h(x_0) \subset \mathrm{co} \bigcup_{\zeta \in I(x_0)} \partial c_F(\zeta, x_0) \tag{4.6}$$

where

$$I(x_0) = \left\{ \zeta \in \hat{\zeta} + MD_B^+ : \ c_F(\zeta, x_0) - \zeta^t y_0 = h(x_0) = 0 \right\}.$$

Combining (4.5), (4.6) and the known Caratheodory Theorem (which says that the convex hull of a set $K \subset X$ is the set of all convex combinations of not more than $1+\dim X$ points of K) we obtain the first conclusion of Theorem 4.1.

To prove the second one let $u_j \in \partial c_F(\zeta_j, x_0)$ $(j=1,2,\dots,k)$, $u \in \partial c_G(\mu, x_0)$ and $v \in N_C(x_0)$ be such that

$$0=\sum_{j=1}^{k} u_j + u + v. \tag{4.7}$$

(This is possible by condition (4.4).) Now let us take an arbitrary point $x \in Q$ i.e., $x \in C$ and $G(x) \cap -E \neq \emptyset$. Then

$$c_G(\mu, x) \leqslant 0. \tag{4.8}$$

By the invexity property there exists $\eta \in T_C(x)$ such that

$$c_F(\zeta_j, x) - c_F(\zeta_j, x_0) \geqslant u_j^t \eta (j=1,2,\dots,k),$$
$$c_G(\mu, x) - c_G(\mu, x_0) \geqslant u^t \eta.$$

Summing up these inequalities and taking account of (4.7) we get

$$\sum_{j=1}^{k} \left[c_F(\zeta_j, x) - c_F(\zeta_j, x_0)\right] + c_G(\mu, x) - c_G(\mu, x_0) \geqslant -v^t \eta \geqslant 0.$$

(The second of these inequalities holds since $v \in N_C(x_0)$ and $\eta \in T_C(x_0)$.) From this and (4.8) it follows that

$$\sum_{j=1}^{k} c_F(\zeta_j, x) - \sum_{j=1}^{k} \zeta_j^t y_0 \geqslant -c_G(\mu, x) \geqslant 0$$

since $\zeta_j \in D_0^{+i}$ and $\mu \in E_0^{+}$.

Setting $\zeta = \sum_{j=1}^{k} \zeta_j$ and observing that

$$c_F(\zeta, x) \geqslant \sum_{j=1}^{k} c_F(\zeta_j, x)$$

we can claim that

$$c_F(\zeta, x) - \zeta^t y_0 \geqslant 0.$$

Since this is true for arbitrary point $x \in Q$ and since $\zeta \in D_0^{+i}$ we conclude from Corollary 2.2 that (x_0, y_0) is a Hartley properly efficient point of Problem (P). □

In general, the number k appearing in Theorem 4.1 is not equal to 1. It is then natural to ask under which conditions we can have $k=1$. An answer to this question is given in Theorems 4.2 and 4.3 below.

THEOREM 4.2. *In addition to the assumptions of Theorem 4.1 assume that for each $\zeta \in D_0^{+i} c_F(\zeta, \cdot)$ is regular at x_0. If (x_0, y_0) is a Hartley properly efficient point of Problem (P) then there exist $\zeta \in D_0^{+i}$, $\mu \in E_0^+$ and $\alpha_0 \in R_+$ such that $\alpha_0 + \|\mu\| \neq 0$ and*

$$0 \in \alpha_0 \partial c_F(\zeta, x_0) + \partial c_G(\mu, x_0) + N_C(x_0). \tag{4.9}$$

The converse statement is true if $\alpha_0 = 1$ and if $c_F(\zeta, \cdot)$ and $c_G(\mu, \cdot)$ are invex on C at x_0.

Proof. Theorem 4.1 yields (4.4) with suitable $\alpha_0 \in R_+$, $\zeta_j \in D_0^{+i} (j = 1, 2, \ldots, k \leqslant 1 + \dim X)$ and $\mu \in E_0^+$.

Setting

$$\zeta = \sum_{j=1}^{k} \zeta_j$$

and noting from [23, Lemma 3.1] that

$$\sum_{j=1}^{k} \partial c_F(\zeta_j, x_0) \subset \partial c_F(\zeta, x_0)$$

we derive (4.9), as desired. The first conclusion of Theorem 4.2 is established. The second one is a direct consequence of Theorem 4.1. □

THEOREM 4.3. *Let F be single-valued. Let $x_0 \in C$ and $z_0 \in G(x_0) \cap -E$. Let set-valued map (3.3) be closed. If x_0 is a Hartley properly efficient point of Problem (P) then there exist $\zeta \in D^{+i}$, $\mu \in E_0^+$ and $\alpha_0 \in R_+$ such that $\alpha_0 + \|\mu\| \neq 0$ and*

$$0 \in \alpha_0 \partial\, \zeta^t F(x_0) + \partial\, c_G(\mu, x_0) + N_C(x_0). \tag{4.10}$$

The converse statement is true if $\alpha_0 = 1$ and if functions $c_G(\mu, \cdot)$ and $\zeta^t F(\cdot)$ are invex on C at x_0.

Proof. The second part of Theorem 4.3 is derived from Theorem 4.1. To prove the first one observe that function f defined in Corollary 2.4 must attain its minimum on Q at x_0. To obtain (4.10) it remains to apply Theorem 3.1 with $\gamma = M$ and $h = F$. (Observe that $D_0^{+i} = D^{+i}$ if F is single-valued.) □

COROLLARY 4.1. *Let F be single-valued, with components $F^1, F^2, \dots, F^m$. Let $x_0 \in C$ and $z_0 \in G(x_0) \cap -E$. Let set-valued map (3.3) be closed. If x_0 is a Hartley properly efficient point of Problem (P) then there exist points $\zeta = (\zeta^1, \zeta^2, \dots, \zeta^m) \in D^{+i}$, $\mu \in E_0^+$ and $\alpha_0 \in R_+$ such that $\alpha_0 + \|\mu\| \neq 0$ and*

$$0 \in \alpha_0 \sum_{j=1}^{m} \zeta^j \partial F^j(x_0) + \partial c_G(\mu, x_0) + N_C(x_0).$$

The converse statement is true if $\alpha_0 = 1$ and if functions $c_G(\mu, \cdot)$ and $F^j(\cdot)(j = 1, 2, \dots, m)$ are invex on C at x_0.

Proof. The sufficiency part is proven by an argument similar to that used in the proof of the sufficiency part of Theorem 4.1. The necessity part can be established by using Theorem 4.3. Indeed, by Theorem 4.3 there exist $\zeta \in D^{+i}, \mu \in E_0^+$ and $\alpha_0 \in R_+$ such that $\alpha_0 + \|\mu\| \neq 0$ and (4.10) holds. To complete our proof it remains to observe from Lemma 1.1 that

$$\partial \zeta^t F(x_0) \subset \sum_{j=1}^{m} \zeta^j \partial F^j(x_0).$$ □

REMARK 4.2. *If $D = R_+^m$, then the Hartley proper efficiency coincides with the Geoffrion proper efficiency (see [13]). In this case $\zeta = (\zeta^1, \zeta^2, \dots, \zeta^m) \in D^{+i}$ means that $\zeta^j > 0$ for all $j = 1, 2, \dots, m$. Thus the necessary condition given in Corollary 4.1 generalizes necessary condition for Geoffrion proper efficiency given in [24] for a special case of Problem (P) where F and G are single-valued and satisfy some invexity assumptions. The arguments used in [24] are based on an alternative result.*

COROLLARY 4.2. *Let F (resp. G) be single-valued, with components $F^1, F^2, \dots, F^m$ (resp. $G^1, G^2, \dots, G^l$). If x_0 is a Hartley properly efficient point of Problem (P) then there exist $\zeta = (\zeta^1, \zeta^2, \dots, \zeta^m) \in D^{+i}$, $\mu = (\mu^1, \mu^2, \dots, \mu^l) \in E^+$ and $\alpha_0 \in R_+$ such that $\alpha_0 + \|\mu\| \neq 0$ and*

$$0 \in \alpha_0 \sum_{j=1}^{m} \zeta^j \partial F^j(x_0) + \sum_{j=1}^{l} \mu^j \partial G^j(x_0) + N_C(x_0),$$

$$\mu^t G(x_0) = 0 \quad (j = 1, 2, \dots, l).$$

Proof. This is a consequence of Corollary 4.1. (Observe that in our case $E_0^+ = \{\mu \in E^+ : \mu^t G(x_0) = 0\}$.) □

REMARK 4.3. *If $C = X$ and $E = \{0\} \times R_+^s \subset R^p \times R^s (p + s = l)$ then condition (CQ) for Problem (P) considered in Corollary 4.2 is assured by the Mangasarian-Fromovitz condition (see Section 3).*

5. Examples

We conclude our paper by some examples.

EXAMPLE 5.1. Let us consider Problem (P) under the following assumptions: $X = R, Y = R^2, Z = R, D = R_+^2, C = R, E = \{0\} \subset R, G(x) = \{x(x-1)\} \subset R, F(x) = \{(-x, x^2 + \frac{1}{2}|x| + \xi) \colon \ \xi \in [0, x^3]\} \subset R^2, x \in R$.

Obviously, for $\zeta = (\zeta^1, \zeta^2) \in D^+ = R_+^2$ we have

$$c_F(\zeta, x) = -\zeta^1 x + \zeta^2 \left(x^2 + \tfrac{1}{2}|x|\right) + \zeta^2 \min(0, x^3).$$

Let us set $x_0 = 0 \in R, y_0 = (0, 0) \in R^2, z_0 = 0 \in R$. Then it is easy to see that (x_0, y_0) is a Hartley properly efficient point of Problem (P), and all the assumptions of Theorem 4.1 are satisfied. Therefore, Theorem 4.1 can be applied while the corresponding conditions for proper efficiency in [11, Theorem 3.3 and Proposition 3.2] cannot be used since the requirement of [11] that E has a nonempty interior is not satisfied in our example.

EXAMPLE 5.2. Let $\mathbb{X}$ be a finite-dimensional Euclidean space and let $G_l : \mathbb{X} \rightrightarrows \mathbb{X} (l = 0, 1, \dots, N-1)$ be locally Lipschitz set-valued maps with nonempty convex compact values where N is a fixed positive integer. Consider the following discrete time dynamical system

$$x(l) \in G_{l-1}(x(l-1)) \quad (l = 1, 2, \dots, N) \tag{5.1}$$

where $x(l) \in \mathbb{X} (l = 0, 1, \dots, N)$ is interpreted as the state of this system at the l^{th} time. A sequence of points $(x(0), x(1), \dots, x(N))$ satisfying (5.1) is said to be a trajectory of system (5.1). Denote by Q' the reachability set of system (5.1), i.e., Q' is the set of all points $\xi \in \mathbb{X}$ such that there exists a trajectory $(x(0), x(1), \dots, x(N))$ of system (5.1) with $x(N) = \xi$.

Let $D \neq \{0\}$ be a closed convex pointed cone of a finite-dimensional Euclidean space Y and let $H : \mathbb{X} \rightrightarrows Y$ be a locally Lipschitz set-valued map with nonempty convex compact values. Consider the following multiobjective optimization Problem (P′):

$$\begin{array}{ll}\text{minimize} & \mathrm{H(x(N))}\\ \text{subject to} & \mathrm{x(l)} \in \mathbb{X}(\mathrm{l}=0,1,\ldots,\mathrm{N})\\ & x(l)\in G_{l-1}(x(l-1))(l=1,2,\ldots,N).\end{array}$$

Let $(x_0(0), x_0(1), \ldots, x_0(N))$ be a trajectory of system (5.1) and let $y_0 \in H(x_0(N))$. We say that $(x_0(0), x_0(1), \ldots, x_0(N), y_0)$ is a Hartley properly efficient point of (P′) if y_0 is a Hartley properly efficient point of the set $H(Q')$ where Q' is the reachability set of system (5.1). We are interested in necessary conditions for $(x_0(0), x_0(1), \ldots, x_0(N), y_0)$ to be a Hartley properly efficient point of (P′).

Let us set $X = \mathbb{X}^{N+1}$, $Z = \mathbb{X}^N$, and let us define set-valued maps $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Z$ by setting

$$F(x) = H(x(N)),$$
$$G(x) = \prod_{l=1}^{N}[-x(l) + G_{l-1}(x(l-1))],$$

where $x = (x(0), x(1), \ldots, x(N))$ is a vector of X with components $x(l) \in \mathbb{X}(l = 0, 1, \ldots, N)$. Then F (resp. G) is a locally Lipschitz set-valued map from X to Y (resp. Z) and has nonempty convex compact values. Using these set-valued maps F and G we see that Problem (P′) can be interpreted as Problem (P) with $C = X = \mathbb{X}^{N+1}$ and $E = \{0\} \subset Z = \mathbb{X}^N$. Since int $E = \emptyset$, it is clear that the approach proposed in [11] cannot be applied to this problem. Now let us consider what we can obtain from Theorem 4.1. Since $E = \{0\} \subset Z$, it is clear that $E^+ = Z$ and the point $z_0 := 0 \in Z$ is the unique point satisfying condition $z_0 \in G(x_0) \cap -E$ mentioned in Theorem 4.1 where $x_0 := (x_0(0), x_0(1), \ldots, x_0(N))$.

For $\mu = (\mu(0), \mu(1), \ldots, \mu(N-1)) \in Z := \mathbb{X}^N$ and $x = (x(0), x(1), \ldots, x(N)) \in X = \mathbb{X}^{N+1}$ we have

$$c_G(\mu, x) = \sum_{l=1}^{N}[-\mu(l-1)^t x(l) + c_{G_{l-1}}(\mu(l-1), x(l-1))]$$

where

$$c_{G_l}(\mu(l), x(l)) := \min_{\xi \in G_l(x(l))} \mu(l)^t \xi \;\; (l = 0, 1, \ldots, N-1).$$

Observe that $(x_0(0), x_0(1), \ldots, x_0(N))$ satisfies (5.1) since it is a trajectory of system (5.1). From this it is clear that $\mu = (\mu(0), \mu(1), \ldots, \mu(N-1)) \in E_0^+$ if and only if

$$c_{G_l}(\mu(l), x_0(l)) = \mu(l)^t x_0(l+1) \;\; (l = 0, 1, \ldots, N-1).$$

We now prove that condition (CQ) is automatically satisfied. Indeed, for $x_0 = (x_0(0), x_0(1), \dots, x_0(N))$ and $\mu = (\mu(0), \mu(1), \dots, \mu(N-1))$ we have

$$\begin{aligned}\partial c_G(\mu, x_0) = {} & [\partial c_{G_0}(\mu(0), x_0(0))] \times [-\mu(0) + \partial c_{G_1}(\mu(1), x_0(1))] \\ & \times \cdots \times [-\mu(N-2) + \partial c_{G_{N-1}}(\mu(N-1), x_0(N-1))] \times [-\mu(N-1)].\end{aligned}$$

From this it is clear that $0 \in \partial c_G(\mu, x_0)$ if and only if $\mu(l) = 0$ $(l = N-1, N-2, \dots, 0)$. This proves that condition (CQ) holds.

We now assume that the following set-valued maps are closed:

$$(\zeta, \xi) \in D^+ \times \mathbb{X} \longmapsto \partial c_H(\zeta, \xi) \subset \mathbb{X}, \tag{5.2}$$

$$(\mu(l), \xi) \in \mathbb{X} \times \mathbb{X} \longmapsto \partial c_{G_l}(\mu(l), \xi) \subset \mathbb{X} \quad (l = 0, 1, \dots, N-1) \tag{5.3}$$

where

$$c_H(\zeta, \xi) = \min_{y \in H(\xi)} \zeta^t y.$$

Applying Theorem 4.1 (and Remark 4.1) we obtain the following result: If $(x_0(0), x_0(1), \dots, x_0(N), y_0)$ is a Hartley properly efficient point of multiobjective optimization Problem (P′) then there exist a positive integer $k \leqslant 1 + (N+1)\dim\mathbb{X}$, points $\zeta_j \in D^{+i} (j = 1, 2, \dots, k)$ and $\mu(l) \in \mathbb{X}(l = 0, 1, \dots, N-1)$ such that

$$\left.\begin{aligned} & 0 \in \partial c_{G_0}(\mu(0), x_0(0)), \\ & \mu(l-1) \in \partial c_{G_l}(\mu(l), x_0(l)) \quad (l = 1, 2, \dots, N-1), \\ & \mu(N-1) \in \textstyle\sum_{j=1}^{k} \partial c_H(\zeta_j, x_0(N)) \end{aligned}\right\} \tag{5.4}$$

and

$$c_H(\zeta_j, x_0(N)) = \zeta_j^t y_0 \quad (j = 1, 2, \dots, k), \tag{5.5}$$

$$c_{G_l}(\mu(l), x_0(l)) = \mu(l)^t x_0(l+1) \quad (l = 0, 1, \dots, N-1). \tag{5.6}$$

Observe that (5.4) is derived from (4.4). Observe also that (5.5) (resp. (5.6)) holds since $\zeta_j \in D_0^{+i}$ (resp. $(\mu(0), \mu(1), \dots, \mu(N-1)) \in E_0^+$).

The above result is obtained under the assumption of the closedness of set-valued maps (5.2) and (5.3). Under some extra conditions this assumption will be satisfied. For example, from [8, Example 2.2] we see that the closedness of set-valued maps (5.3) is satisfied if

$$G_l(\xi) = \{g_l(\xi, u) : u \in U\} \quad (l = 0, 1, \dots, N-1) \tag{5.7}$$

where U is a compact subset of an Euclidean space $\mathcal{U}$ and $g_l : \mathbb{X} \times \mathcal{U} \to \mathbb{X}$ is a single-valued map of the class C^1. Similar sufficient conditions for the closedness of set-valued map (5.2) can be formulated.

Observe that under assumption (5.7) system (5.1) can be seen as a discrete time dynamical system whose behavior is described by the following equations

$$x(l) = g_{l-1}(x(l-1), u(l-1)) \ \ (l = 1, 2, \dots, N)$$

where the "control" $u(l)$ of the system at l th time is subject to condition $u(l) \in U (l = 0, 1, \dots, N-1)$. This system is often encountered in the optimal control theory for discrete time dynamical systems. We refer the reader to [15, Section 4] for necessary conditions for Geoffrion proper efficiency in a multiobjective discrete time optimal control problem involving not only controls but also parameters.

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